Valuing Options: A Simplified Approach

In Chapter 13 we specified the value of an option at maturity. However, the question facing the buyer and seller of the option is not the value at maturity but rather the value today. This question bedeviled the finance profession from the time option pricing was first addressed in 1900 until the question was finally successfully answered in 1973 by Fischer Black and Myron Scholes. The Black-Scholes option pricing model, the topic of discussion in the next chapter, is of necessity somewhat mathematical. So let's first look at a simplified approach to option pricing that was provided by William F. Sharpe and expanded by John C. Cox, Stephen A. Ross, and Mark Rubinstein. This approach is referred to as the binomial option pricing model. We will present the binomial pricing model by looking at the pricing of an European-style option. Once this is complete, we will look briefly at the pricing of an American-style option.

1. Among the first to address the question of option pricing was Louis Bachelier in his dissertation Théorie de la Spéculation at the Sorbonne in 1900.
5. Alternatively, the model is referred to as the Cox, Ross, Rubinstein (C-R-R) Approach.
The Binomial Pricing Model

In this simplified approach to option pricing, we look at the continuous-process world as a series of snapshots—in much the same way that a movie is a series of still pictures. Let’s suppose that today, on day 0, the price of a particular share of stock is $100. Let’s suppose further that tomorrow it could rise or fall by 5%:

<table>
<thead>
<tr>
<th>Day 0</th>
<th>Day 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>105   Up by 5%</td>
</tr>
<tr>
<td></td>
<td>95    Down by 5%</td>
</tr>
</tbody>
</table>

Aside

At first glance, the preceding model looks too simple to have any connection to the real world. However, it is simple only because we have let the share price move only once per day. How about if we let the share price move up or down by 5% every twelve hours,

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>105</td>
</tr>
<tr>
<td></td>
<td>95</td>
</tr>
<tr>
<td></td>
<td>110.25</td>
</tr>
<tr>
<td></td>
<td>99.75</td>
</tr>
<tr>
<td></td>
<td>90.25</td>
</tr>
</tbody>
</table>

or every six hours, or....

If we make the intervals shorter, this simple binomial model becomes more realistic.

Let’s consider a one-day call option on this share of stock. For simplicity, let’s set the exercise price of this one-day call option at $100. Hence, the value of the call option at maturity—day 1—is:

<table>
<thead>
<tr>
<th>Share Price at Expiration</th>
<th>Value of Call at Expiration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$105</td>
<td>$5</td>
</tr>
<tr>
<td>95</td>
<td>0</td>
</tr>
</tbody>
</table>

However, a party thinking about selling (or buying) this call option is not as interested in its value at maturity as in its value today. A number
of very smart people have tried to find this value directly... and failed. The approach we are going to follow is modeled on the approach of Black and Scholes. We will value the option by valuing an arbitrage portfolio that contains the option.

An arbitrage portfolio is one that earns a riskless return. The object is to form such a portfolio out of the two risky assets—the share of stock and the call option on the share of stock—so that the gains made on one of the assets would be exactly offset by losses on the other. As the following table indicates, for the case in point, such a portfolio could be formed by creating a portfolio that is long one share of stock and short two call options:

<table>
<thead>
<tr>
<th>S</th>
<th>C</th>
<th>2C</th>
<th>3−2C</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>5</td>
<td>10</td>
<td>95</td>
</tr>
<tr>
<td>95</td>
<td>0</td>
<td>0</td>
<td>95</td>
</tr>
</tbody>
</table>

As indicated, the value of the portfolio—$S - 2C$—will be $95, regardless of the value of the share of stock. Hence, we have formed a portfolio that has no risk.

Aside

Calculating the hedge ratio

In the preceding illustration, it was pretty easy to see that two calls would exactly hedge the move in one share of stock. However, we need a more general rule when we encounter more complex situations. Not surprisingly, such a rule involves nothing more than a little algebra. For the case in point, we want to find the number of call options that will make the value of the portfolio when the share price is $105$,

$$105 - N \times S$$

equal to the value of the portfolio when the share price is $95$,

$$95 - N \times 0$$

Hence, if

$$105 - N \times S = 95 - N \times 0$$

it follows that $N = 2$.

Generalizing, let’s follow Cox, Ross, and Rubenstein by defining the high share price as $U = S$ and the low share price as $L = dS$. For the case
we are examining, \( u = 1.05 \) and \( d = 0.95 \). Continuing to follow Cox, Ross, and Rubenstein, we define the value of the call option when share price is high as \( CU = C(uS, T) \) and the value of the call when share price is low as \( CD = C(dS, T) \). Then, the number of calls necessary to form the arbitrage portfolio \( (N) \) is

\[
N = \frac{(SH - SL)(CU - CD)}{(uS - dS)(CU - CD)} = \frac{(SH - SL)}{(uS - dS)} = \frac{[uS - dS](C(uS, T) - C(dS, T))}{(SH - SL)}
\]

In the remainder of this discussion, we will refer to the hedge ratio \( (\Delta) \) as the inverse of the number of calls necessary to form the arbitrage portfolio.

\[ \Delta = \frac{1}{N} \]

Hence, for the case in point, the hedge ratio is \( \Delta = 1/2 \).

On day 0, the value of the share is $100 so we can express the value of the arbitrage portfolio, \( S - 2C \), as \( 100 - 2C \). What we want to find out is the value of the call option, \( C \), on day 0—a value that is so far unknown. We do know, however, that on day 1, the value of the portfolio is 95. Hence, we know that

\[ (100 - 2C)_{\text{day 0}} \rightarrow (95)_{\text{day 1}} \]

In order to turn this into an equality and thus solve for \( C \), the value of the call option on day 0, it is necessary to take the present value of the $95 to be received in one day:

\[ 100 - 2C = 95/(1 + r) \]

where \( r \) is a one-day interest rate. Since the arbitrage portfolio is riskless, the interest rate used is the risk-free interest rate. Continuing our example, if the annualized one-day risk-free rate—the treasury bill rate for one-day bills—is 7.5%, the rate for one day is \( (1/365) \times 0.075 = 0.0002 \) and the preceding equation becomes

\[ 100 - 2C = 95/(1.0002) \]

Hence, for our example, the value of the one-day call option on day 0 with a prevailing share price of $100 is $2.54 (see Figure 14-1).
An option payoff diagram

The tree diagram in Figure 14-4 provides the value of the call for a particular share price on day 0. In this instance, a share price of $100. To transform this tree diagram into a more familiar option diagram, we need to think about the value of the call on day 0 for various share prices.

To accomplish this, we need the general expression for the value of this one-period call option as provided by Cox, Ross, and Rubinstein:

\[
C = \frac{[uL + \sigma - d]CU + [u - \sigma + \sigma L]CD}{(1 + r)}
\]

Using this general equation, consider these cases:

1. If on day 0, the share price is less than or equal to X/2, the value of the call option is zero. For this case to point, if the share price on day 0 is less than or equal to 95.73, the value of the call is zero.

   on day 0, the share price is greater than or equal to 104, the value of the call is S - X(1 + r) - x. For the case in point, if the share price on day 0 is greater than or equal to 105.26, the value of the call is X.

2. If the share price is greater than X/2 but less than X, the call option is:

   \[
   \frac{[uL + \sigma - d]CS - X}{(1 + r)}
   \]

---

For Rubinstein, Options: Market Approach
For the one-period option we have been looking at:

<table>
<thead>
<tr>
<th>Share Price on Day 0</th>
<th>Value of Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>3.04</td>
</tr>
<tr>
<td>100</td>
<td>2.51</td>
</tr>
<tr>
<td>99</td>
<td>1.98</td>
</tr>
</tbody>
</table>

The payoff diagram for the specific option we have been examining can then be drawn as follows:

Or, more generally, the payoff diagram for a one-period option can be illustrated as:

At this point, you may be saying that this is all very well and good, but not very relevant since it is unlikely we will encounter many one-
day option contracts. So let’s see what happens when we let the option run for two days. Continuing to assume that the share price can move up or down by 5% every day, the distribution of share prices over the three days and the value of the call option at expiration are as follows:

<table>
<thead>
<tr>
<th>Day 0</th>
<th>Day 1</th>
<th>Day 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>105</td>
<td>110.25</td>
</tr>
<tr>
<td>95</td>
<td>99.75</td>
<td>10.25</td>
</tr>
</tbody>
</table>

Again, we want to know the value of the call option at day 0. To find this out, we must first determine the values of the option on day 1 and then use these values to determine what the option will be worth on day 0:

**Day 1**

*If the value of the share is 105: The arbitrage portfolio would be $S = (1.025)C$. That is, the number of shares necessary to hedge one share of stock would be*

\[
N = \frac{(110.25 - 99.75)}{(10.25 - 0)} = 10.50/10.25 = 1.025
\]

(The hedge ratio is \( 1/N = 0.976 \).) So,

\[ (105 - 1.025C)_{\text{day 1}} = (99.75)_{\text{day 2}} \]

Therefore,

\[ (105 - 1.025C) = (99.75)/(1.0002) \]

*The value of the call would be $5.14.*

*If the value of the share is 95, the value of the call is 0.* The value of the call will be zero regardless of whether the value of the share rises to 99.75 or falls to 90.25.

8. Put another way, since the number of shares necessary to hedge one share of stock is, in the limit, positive infinity,

\[
N = \frac{(99.75 - 90.25)}{(0 - 0)} = 9.50/0
\]

the value of the call option must approach zero as a limit.
The relevant distribution has become

\[
\begin{align*}
100 & \quad C = 5.14 \\
95 & \quad C = 0
\end{align*}
\]

Hence, the number of call options needed to hedge one share of stock is

\[
N = \frac{(105 - 95)/(5.14 - 0)}{10/5.14} = 1.95
\]

(The hedge ratio is 0.514.) Thus, the arbitrage portfolio is \( S - 1.95C \):

\[
(100 - 1.95C)_{\text{day 0}} \rightarrow (95)_{\text{day 1}}
\]

Continuing to use 7.5% as the relevant annualized rate for a one-day treasury bill,

\[
(100 - 1.95C) = 95/1.0002
\]

The value of the call option would be 2.579—i.e., $2.58.

This valuation is summarized in Figure 14-2. And if we can value a two-day option, we can value a three-day, four-day, or n-day option. The logic is exactly the same: we solve iteratively from expiration to time period 0. The only thing that changes is the size of the problem.

**Figure 14-2. Valuing a Two-Day Option.**

<table>
<thead>
<tr>
<th>Day 0</th>
<th>Day 1</th>
<th>Day 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>99.75</td>
<td>90.25</td>
</tr>
<tr>
<td>$ C = 5.14 \quad \Delta = 0.514 $</td>
<td>$ C = 0.0 \quad \Delta = 3.0 $</td>
<td>$ C = 10.25 \quad \Delta = 0.976 $</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
105 & \quad C = 2.58 \\
95 & \quad C = 0.0
\end{align*}
\]
The preceding discussion has had two objectives. The first is obvious: to demonstrate to you that the pricing of an option is not as difficult as it might otherwise seem. The second objective of this discussion of pricing is much more subtle: to show you the five variables that determine the value of an option. To see these variables, look again at the examples.

The variables we employed to value the option were:

*The prevailing share price, S.* In our example, the share price on day 0, the date of origination of the option contract, was $100.

*The exercise price of the option, X.* In our example, we used $100 as the exercise price.

*The time to expiration of the option, T.* We considered both one day and two days.

*The risk-free interest rate corresponding to the time remaining on the option, r.* In our example, the annualized rate for a one-day T-bill was 7.5%, so the one-day interest rate was 0.0020.

*The volatility in the share price, σ.* In the form of the binomial pricing model we have been using, the value of the call option was determined in part by the magnitude of the movements in the share price. In our example, we have used price movements of 5% up or down per day. This magnitude could be summarized by the variance in the distribution of share prices.

Hence, we could write an implicit function for the value of a call option as

\[ C = C(S, X, T, r, \sigma) \]  

(14-1)

where the following relationships hold:

- Increases in the share price increase the value of the call option.
- If we increase the original share price by $10 in our example.

<table>
<thead>
<tr>
<th>Share Price</th>
<th>Change</th>
<th>Value Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>115.50</td>
<td>Up 5%</td>
<td>2.50</td>
</tr>
<tr>
<td>104.50</td>
<td>Down 5%</td>
<td>-2.50</td>
</tr>
</tbody>
</table>

while leaving the exercise price at $100 and the other determinants unchanged, the value of the call option will rise from $2.51 to $5.52.
Increases in the exercise price decrease the value of the call option. If we change our example by increasing the exercise price from $100 to $101, the value of the one-day call option would fall from $2.51 to $2.01.

Increases in the time to expiration increase the value of the call option. As noted, increasing the time to maturity from one to two days increased the value of the call option from $2.51 to $2.58.

Increases in the risk-free interest rate increase the value of the call option. If the annualized rate on a one-day T-bill rose from 7.5% to 15%, the daily risk-free interest rate would rise from 0.0002 to 0.0004, and the value of the one-day call option would rise from $2.51 to $2.52.

Increases in the volatility of share price increase the value of the call option. Suppose that instead of 5% up or down each day, share prices could move up or down by 10% each day:

\[ 110 \quad \text{Up 10\%} \]
\[ 90 \quad \text{Down 10\%} \]

With no other changes in the determinants of the option value, the value of our one-day call option would rise from $2.51 to $5.01.

The preceding relationships can be summarized as

\[ C = C(S, X, T, r, \sigma) \]

Then, using the put-call parity relations defined in the preceding section, the value of a European-style put option can be expressed as:

\[ P = C - S + XD = P(S, X, T, r, \sigma) \]

From the definition of a put option it follows that the right to sell an asset at price \( X \) becomes more valuable as the market price of the asset \( S \) falls or the exercise price \( X \) rises. From the put-call parity relation, the effect of \( T \) on the value of a put is indeterminate because it increases the value of the call \( C \) but decreases the discounted exercise
price (XD). In the case of \( r \), the decrease in the discounted exercise price exceeds the increase in the value of the call; so, an increase in \( r \) decreases the value of the put. Finally, since an increase in volatility \( (\sigma) \) increases \( C \) but has no effect on \( S \) or \( XD \), it follows directly that an increase in volatility will increase the value of the put option.

**A Note on American Options**

An American option gives its owner all of the rights contained in a European option plus the right to *early exercise*; that is, the owner of the American option has the right to exercise the option prior to maturity. It follows, then, that the value of the American option is always at least that of the European option. Whether the American option is worth more than the European option depends on whether or not the option would ever be exercised early.

Hence, the question of the value of an American option depends on yet another question: Will an American option ever be exercised early? As with so many other questions, the answer is: *it depends.*

**American Calls**

If the share of stock pays no dividend, it is never optimal to exercise the American call option early. Suppose that, at some time prior to expiration, the American call option is "in the money"; that is, the prevailing share price \( (S) \) is greater than the exercise price for the call option \( (X) \). If the option is exercised, the owner will receive a gain equal to the difference between the prevailing share price and the exercise price for the option. However, if instead the owner of the option sells the option, we know from the preceding discussion that the market value of the option is equal to the difference between the prevailing share price and the discounted exercise price plus the remaining time value of the option. The strategies are detailed in Table 14-1.

As long as there is time remaining to maturity of the option, \( D < 1 \) (and Time value \( > 0 \)); thus, it follows that \( (S - XD) > (S - X) \) and

\[
S - XD + \text{Time value} > S - X
\]

(14-4)

9. And, consequently, whether it will be necessary to use a valuation model other than the Black-Scholes model or the binomial model outlined in this chapter.
Table 14.1. Strategies for Realizing an In-the-Money American Option.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exercise the option early</td>
<td>$S - X$</td>
</tr>
<tr>
<td>Sell the option</td>
<td>$S - XD + \text{Time value}$</td>
</tr>
</tbody>
</table>

Hence, for a non-dividend-paying stock, early exercise of an American call option will never occur. (Therefore, in this case, the value of an American call option is identical to that of a European call option.)

For an American call option on a dividend-paying share of stock, early exercise will be optimal if the dividend is sufficiently large; and early exercise will occur on the last day before the ex-dividend day, if at all. Since shares purchased on or after the ex-dividend day do not receive the next dividend, the share price will fall on the ex-dividend day by an amount equal to the dividend. It is this drop in price that can provide the incentive for early exercise of the American call option. The party who exercises the option just before the ex-dividend date will receive a dividend payment that will not be received if the option is not exercised. Hence, the question becomes whether the amount received from exercising the option,

$$S - X + \text{Dividend}$$

is greater or less than the value of the call option if it were to be sold:

$$S - XD + \text{Time value}$$

It follows that the American call option would be exercised early if

$$\text{Dividend} > X (1 - D) + \text{Time value} \quad (14.5)$$

To value the American call option, the value of this early exercise provision would have to be determined for each of the ex-dividend dates that occur during the life of the option.

American Puts

Early exercise of an American put option will be optimal if the price of the stock falls sufficiently below the exercise price. This rather complex concept is most easily seen via an example. Let's consider an American
put option that has the following characteristics:

\[ X = 100 \]
\[ T = 1 \text{ year} \]
\[ r = 20\% \]

Suppose the price of the share falls to $10. The person exercising the option early will receive \( X - S = 100 - 10 = 90 \) today. And if that $90 is held in a T-bill, he or she will have at maturity \( 90 \times (1.2) = 108 \). On the other hand, if the option is held to maturity, the most the option would be worth is $100—and this only if the share price fell to zero. In this case, since $108 > $100, it is clear that the American put option would be exercised early.

If, however, the share price only falls to $20, the situation is more complex. Early exercise of the option will bring $80 today and $80 \times (1.2) = 96$ at the end of one year. As before, the option could be worth at maturity as much as $100 so it may not be optimal to exercise early. However, the option is worth $100 only if the value of the share drops to zero. If instead the share price falls to $5 at expiration, the value of the option would be $95, and it would be optimal to exercise the option early. Hence, in this case, it may or may not be optimal to exercise the American put option early, depending on the probability distribution of share prices.

The point is that to determine the value of an American put option on a non-dividend-paying stock, it is necessary to determine whether it would be optimal to exercise the option early on any of the days prior to expiration. Since there is no closed-form equation that will provide the solution, this involves a large, iterative, numerical approximation problem: we first check to see if early exercise could be optimal on the day prior to expiration, then on the day before that, then on the day before that, and so on.

Interestingly, while dividends make it more difficult to value American call options (because they make it possible for early exercise to be optimal), the existence of dividends makes it less difficult to value American put options. As illustrated, American puts are exercised early only if there is a sufficiently large drop in the share price. For non-dividend-paying stocks, we would have to check this relation for each
trading day prior to expiration. However, for dividend-paying stocks, the predictable share price drop on the ex-dividend dates makes the probability of optimal early exercise highest on ex-dividend dates. Hence, for American put options on dividend paying stocks, the value can be closely approximated by considering early exercise only for the remaining ex-dividend dates.