In Chapter 9 we demonstrated that a swap can be decomposed into either a portfolio of loans or a portfolio of forward contracts. In Chapter 12 we will make use of the concept of a swap as a portfolio of forwards to gain insights into the default risk of a swap. In this chapter, we use the concept of a swap as a portfolio of loans to provide insights into the pricing of a swap.

Pricing an At-Market Swap

Figure 11-1 again illustrates the equivalence of an interest rate swap and a pair of loans. The implication of this figure is that if you can value (price) loans, you should be able to value a swap contract. Put another way, if you know the mechanics of pricing loans, you should be able to determine the appropriate fixed rate in the swap illustrated in Figure 11-1.

And that is indeed the case. The loans are both zero-expected-NPV (net present value) projects. Consequently, since the swap is nothing more than a long and a short position in loans, the expected NPV of the swap must also be zero. Hence, if the actual or expected floating-rate payments at time periods $1, 2, \ldots, T$ can be determined and if the term structure of interest rates is known, the NPV of the swap can be set equal to zero, and we can solve for the fixed rate. Perhaps the best way to explain this is to go directly to an example.

1. We again presume here that the capital markets are efficient.
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Figure 11.1. Decomposition of an Interest Rate Swap into a Portfolio of Loans.

Interest rate swap

\[ \bar{R}_1 \rightarrow \bar{R}_2 \rightarrow \bar{R}_7 \rightarrow \bar{R}_T \]

Borrowing:
fixed rate

\[ \bar{R}_1 \rightarrow \bar{R}_2 \rightarrow F \rightarrow \bar{R}_T \]

Lending:
floating rate

\[ \bar{R}_1 \rightarrow \bar{R}_2 \rightarrow \bar{R}_T \rightarrow F \]

The cash flows from an interest rate swap where the party pays fixed is equivalent to the cash flows of a portfolio of two loan contracts, where borrowing is at a T-period fixed rate ($\bar{R}_T$), lending is at a floating rate ($\bar{R}_f$), and $F$ is the face value of the loan.

Example

Pricing an interest rate swap

Galactic Industries (GI) wishes to enter into a swap in which GI will pay cash flows based on a floating rate and receive cash flows based on a fixed rate. In the jargon of the swap market, Galactic—the floating rate payor—is referred to as the seller of the swap (or as being short the swap).

Market convention is to quote the terms of interest rate swaps as the floating-rate index (normally LIBOR) that against some fixed rate; that is, Galactic will pay cash flows based on the floating rate flat and will receive cash flows based on a fixed rate of 5%. The question is: “What is the appropriate fixed rate—what’s X?”

To keep our calculations at a minimum, suppose GI requested a quote from the Dead Solid Perfect Bank (DSPB) on the following simple swap:
Notional principal amount: \$100
Maturity: One year
Floating index: Six-month LIBOR
Fixed coupon: 5% (30/360^2)
Payment frequency: Semiannual

From these terms, we know what Galactic will pay: At the six-month settlement, GI pays a "coupon" determined by the six-month LIBOR rate in effect at contract origination. At the twelve-month settlement, GI's "coupon" payment is determined by the six-month LIBOR rate prevailing at month 6. What is missing is how much Galactic will receive—how much DSFB will pay.

Suppose that the LIBOR yield curve (the spot yield curve) prevailing at origination of this swap is the simplified yield curve shown below.

\[ \text{8\%} \quad \text{10\%} \]
\[ \text{6 mo.} \quad \text{1 yr.} \quad \text{Time to maturity} \]

To determine the appropriate fixed rate, the managers of DSFB must consider the contractual/expected cash flows from this swap:

\[ \tilde{A_1}, \quad \tilde{A_2} \]
\[ \tilde{F_1}, \quad \tilde{F_2} \]

The first floating-rate inflow, \( \tilde{A}_1 \), is the easy one. The floating-rate cash flow DSFB will receive at the first settlement date is determined by the six-month

2. We use the 30/360 convention for convenience in our example. In truth, the day count convention for LIBOR is Actual/360, as is the case for commercial paper and bankers acceptances.
rate in effect at contract origination: 8%. Hence, at the six-month settlement, DSPB expects to receive

$$\tilde{R}_1 = \$100 \times (180/360)(0.08) = \$100 \times \frac{1}{2}(0.08) = \$4.00$$

Note that in this calculation and all that follow, we use the bond method for calculating interest payments.3

To obtain the expected floating-rate inflow at the one-year settlement, we need to know the six-month rate in six months. As we know from our discussions in Part II, this rate—the rate from \( t = 6 \) months to \( t = 1 \) year—is the forward rate. Arbitrage guarantees that

$$(1 + r_{12}) = \left[1 + \frac{1}{2}(r_{66})\right] \times \left[1 + \frac{1}{2}(r_{61})\right]$$

where \( r_{12} \) and \( r_{66} \) are, respectively, the current twelve-month and six-month zero rates. Using this arbitrage condition, the six-month and one-year rates of

3. For maturities less than one year, prevailing market practice is to quote interest rates with compounding already included in the rate. Hence, if the annualized six-month rate is 8%, the amount that will be received at the end of six months on a $100 investment is

$$\$100 \times (180/360) \times 0.08 = \$4.00$$

using the convention that compounding occurs annually but the periodicity of the rate is monthly.

In contrast, the convention used by most finance textbooks is to treat the interest rate as subject to compounding. The common method of compounding is discrete compounding. Using this method, if the annualized six-month rate is 8%, the amount that will be received at the end of six months on the $100 investment is

$$\$100 \times (1.08)^{1/2} - \$100 = \$3.92$$

where the periodicity is again monthly but the rate is now compounded monthly. Put another way, to yield the $4.00 at the end of six months, the stated interest rate using the method of discrete monthly compounding would be 8.16%:

$$\$100 \times (1.0816)^{1/2} - \$100 = \$4.00$$

Although the different conventions are sometimes confusing, they cause no problem as long as the user knows which convention is being used.

4. In footnote 3, we noted the various ways in which interest rates can be quoted and the coupon calculated. Had the interest rates been quoted subject to monthly compounding, the arbitrage condition would have had to take the compounding into consideration. If we denote the annualized rate subject to compounding as \( r \), the arbitrage condition would become

$$(1 + r_{12}) = (1 + r_6)(1 + r_{61})$$
8% and 10%, respectively, require that the forward rate \( a_{11} \)—the six-month rate in six months—be 11.5%. Therefore,

\[
R_2 = 100 \times \frac{1}{1.115} = 87.56
\]

Hence, the contractual/expected floating-rate inflows to DSPB are as illustrated below.

\[
\begin{array}{cc}
\text{6 mo.} & \text{1 yr.} \\
\$4.00 & \$5.75 \\
\end{array}
\]

What DSPB needs to determine are the outflows; the appropriate fixed-rate payments. In origination, the expected net present value of this in-market swap is zero. That is,

\[
\frac{4.00 - R_1}{1 + \frac{1}{1.10}} + \frac{5.75 - R_2}{1.10} = 0
\]

where \( R_1 = R_3 \), solving this equation, \( R_1 = R_2 = 4.85 \). Hence, the appropriate fixed rate is 5.70%.

Post-looking at the term structure of interest rates, it might seem that the appropriate fixed, one-year interest rate is 10%, in which case the fixed-rate outflows would be

\[
R_1 = R_2 = 100 \times \frac{1}{1.10} = 90.91
\]

However, if \( R_1 = R_2 = 5.00 \),

\[
\begin{array}{cc}
\text{6 mo.} & \text{1 yr.} \\
\$4.00 & \$5.75 \\
\$5.00 & \$5.00 \\
\end{array}
\]

the present value of the swap to DSPB would be negative:

\[
\frac{-1.00}{1 + \frac{1}{1.08}} + \frac{0.75}{1.10} = -0.28
\]

The problem is that 10% is a zero-coupon rate. As should be clear from Figure 11-1, \( R \) is associated with a coupon-bearing instrument (loan). What we need is
Swaps

The market coupon interest rate—the par rate. The par rate is the coupon rate that would put the bond trade at par. In our case, that means the compounded annualized par rate for 1 yr. is given by:

\[ \frac{\frac{\text{par}}{100}}{1 + \left(\frac{0.38}{1.10}\right)}\times 100 = \frac{9.70}{1.38} \]

Solving, the one-year par rate, \( r_{1\text{yr.}} \), is 9.70%, precisely the rate we determined earlier.

Hence, in the case of this simple, at-market swap, the appropriate fixed rate is the one-year par rate—9.70%. The terms of this swap can now be completed:

- **Notional principal amount**: $100
- **Maturity**: One year
- **Floating index**: Six-month LIBOR
- **Fixed coupon**: 9.70%
- **Payment frequency**: Semiannual
- **Day count**: 30/360

The expected cash flows for DSPB are as illustrated below:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$4.00</td>
<td>$5.75</td>
</tr>
<tr>
<td>8%</td>
<td>11.5%</td>
</tr>
</tbody>
</table>

\[ \frac{4.00}{1.10} = 3.64\]  
\[ \frac{4.00}{1.10} = 3.64 \]

As our example illustrates, pricing an interest rate swap requires that the cash flows be identified and then discounted by the zero-coupon (spot) interest rate. To obtain the expected cash flows for the floating payments, it was necessary to obtain the forward interest rate from the forward yield curve. Finally, in the case of this simple, at-market swap, the appropriate fixed rate was simply the par rate. Hence, to price an interest rate swap, we ended up using three yield curves: the zero-coupon yield curve, the forward yield curve, and the par yield curve.

**Swap Pricing Conventions**

The swap described in the preceding example and illustrated in Figure 11-1 is referred to (almost condescendingly) as a plain vanilla interest
rate swap. For this simple one-year swap, we ended up with a quote of six-month LIBOR (the spot rate) against the one-year par rate.

The market convention has come to be to price these plain vanilla swaps as LIBOR flat against the U.S. treasury (par) rate plus. An illustration of market-style quotations for at-market interest rate swaps at origination is provided in Figure 11.2. On Monday, September 12, 1988, the market was pricing a three-year interest rate swap in the interbank market such that:

If you want to receive the fixed rate, you will pay LIBOR and receive the three-year treasury par rate plus 74 basis points.

If you want to pay the fixed rate, you will receive LIBOR and pay the three-year treasury par rate plus 77 basis points.

The difference between the receive treasuries and pay treasuries, 3 basis points, was the bid-ask spread.

Valuing a Swap: Marking the Swap to Market

The market convention of LIBOR versus treasuries plus spread works well if all you want to do is price at-market swaps, at origination. However, this par rate convention does not work if you need to value a swap after origination or if you need to value (price) an off-market swap.

Figure 11.2. U.S. Dollar Interest Rate Swap Quotes

<table>
<thead>
<tr>
<th>U.S. Dollar Rate Swap Quotes (Treasury-LIBOR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 Yr.</td>
</tr>
<tr>
<td>3 Yr.</td>
</tr>
<tr>
<td>4 Yr.</td>
</tr>
<tr>
<td>5 Yr.</td>
</tr>
<tr>
<td>7 Yr.</td>
</tr>
<tr>
<td>10 Yr.</td>
</tr>
</tbody>
</table>

Source: Telebank, Fulton Pribon USA, Inc. (September 12, 1988). All Rights Reserved. Reproduced with permission.
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After origination, the only way to value a swap is to employ the zero-coupon yield curve. Once the swap has been contracted, its value depends on what happens to the market price on which the swap is based. The value of a dollar/sterling currency swap to the party paying dollars rises (falls) as the value of sterling rises (falls). The value of a commodity swap varies with the market price of the commodity. And, as we illustrate in the continuation of our example, the value of an interest rate swap depends on what happens to market interest rates.

Example

Valuing an interest rate swap

Galactic Industries (GI) and the Dead Solid Perfect Bank (DSPB) contracted to the interest rate swap outlined in the preceding example on the afternoon of July 23.

On the morning of July 24, the LIBOR yield curve (zero-coupon curve) shifted up by 1%, as illustrated below.

![Interest Rate Curve]

The terms of the swap contract specified that DSPB will pay at an annual rate of 9.70%. DSPB’s first floating-rate receipt was determined at origination, so the $4.00 DSPB will receive in six months is unchanged. For the one-year swap, the only cash flow that will be changed is the expected floating-rate inflow at one year. With the new term structure, the forward rate, \( r_{11} \), is 12.4%:

\[
(1 + 0.11) = (1 + \frac{1}{2}(0.09))(1 + \frac{1}{2}(0.11))
\]

Thus, the expected floating-rate inflow in one year is

\[
\hat{R}_2 = 100 \times \frac{1}{2}(0.124) = 66.22
\]

and DSPB’s expected cash flows are as illustrated below.
Calculating DSPB’s expected net cash inflows,

\[
\begin{array}{c|c|c}
\text{6 mo.} & \text{1 yr.} \\
\hline 
5.400 & $6.22 \\
(8\%) & (12.4\%) \\
5.485 & $4.85 \\
(9.7\%) & (9.7\%) \\
\end{array}
\]

6 mo. 1 yr.

and discounting these expected net cash inflows by the corresponding zero rates from the current zero-coupon yield curve,

\[
\begin{array}{c|c|c}
\text{5 mo.} & \text{1 yr.} \\
\hline 
-50.85 & +51.37 \\
9\% & 11\% \\
\end{array}
\]

the value of the swap to DSPB has risen from zero at origination to 42 cents:

\[
\frac{-50.85}{1 + \frac{1}{1.09}} = 50.42
\]

In the preceding example we have marked the swap to market. If we calculate the value of the swap for different changes in the yield curve, that is, if we mark the swap to market for different shifts in the yield curve, we can obtain a payoff profile for the swap. For example, if we look at the value of the preceding swap for shifts in the treasury zero curve of +3%, -1%, -2%, and -3%, the average change in the value of the swap contract per 1% change in the yield curve—\(\frac{\text{Value of swap}}{\Delta r}\)—is 42 cents. Hence, we can sketch the payoff profile for this swap as in Figure 11-3.5

The lesson from all this is simple. To price an on-market interest rate swap, you can use the par yield curve. But to value a swap after origination, it is necessary to use the zero-coupon yield curve rather than the par yield curve.

5. Although the value of the swap contract is, in truth, a nonlinear function of interest rates, we continue to use a linear approximation.
Pricing an Off-Market Swap

As with valuing a swap after origination, it is necessary to use the zero-coupon yield curve—not the par yield—to price an off-market swap.

Figure 11-4 illustrates an off-market swap. In the case illustrated, the fixed rate paid by this party is higher than the prevailing market fixed rate: the fixed-rate payor is paying above-market “coupons.” Consequently, at contract origination, a payment will have to be made from the floating-rate payor to this fixed-rate payor. The question is: How large should this initial payment be?

As Figure 11-4 indicates, the size of the initial payment from the floating-rate payor to the fixed-rate payor is determined by the difference between the market value of a bond that carries the above-market interest rate and the notional principal of the swap.

To make this more concrete, let’s look at an example of an off-market swap. Consider a delayed LIBOR reset swap (also called a LIBOR in arrears swap). ⁶

---

Figure 11-4. An Off-Market-Rate Swap.

(a) Off-Market-rate interest rate swap.
\[ (P^P - P) \]
\[ R_1 \]
\[ R_2 \]
\[ R_f \]

(b) Borrowing: fixed rate.
\[ (R - R) \]
\[ P \]
\[ R_0 \]
\[ R_1 \]
\[ R_f \]
\[ P \]

(c) Lending: floating rate.
\[ P \]
\[ R_0 \]
\[ R_1 \]
\[ R_f \]

Dates:
0 1 2 \ldots T

The party pays cash flows \((R)\) determined by a fixed interest rate above the current market rate and receives cash flows \((R_f)\) determined by the relevant floating interest rate. In part a, a principal exchange \((P^P - P)\) occurs at origination, with \(P^P\) equal to the market value of a bond with coupons \(R\), and a principal repayment of \(P\). In part b this swap is decomposed into two loan contracts: borrowing at a fixed rate higher than the prevailing market rate, and lending at the market floating rate.

Example

A delayed LIBOR reset swap

In the normal (plain vanilla) swap, the rate paid at month 6 is the six-month rate in effect at month 0, and the rate paid at month 12 is the six-month rate in effect at month 6.

In a delayed reset or in-arrears swap, the rate paid at month 6 is the six-month rate in effect at month 6 and the rate paid at month 12 is the six-month rate in effect at month 12.

Let's change our swap between Galactic Industries and the Dead Solid Perfect Bank to a delayed LIBOR reset swap. With this structure, at origination, the rate DSPE expects to receive at month 6 is not the six-month spot rate at origination, 8%, but the forward rate—the six-month rate in six months—11.5%. Hence,

\[ R_1 = (\$100)(1.08)^{0.115} = 5.75 \]
Likewise, in an in-arrears swap, the rate DSPB expects to receive at month 6 is not the six-month rate in six months but the six-month rate in twelve months. Let's suppose the forward rate is 13%.

\[ R_6 = (\$100)(0.13) = \$6.50 \]

Hence, DSPB's expected outflows are as illustrated below.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$5.75</td>
<td>$6.50</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>6 mo.</td>
<td>1 yr.</td>
</tr>
</tbody>
</table>

To determine the appropriate fixed rate for DSPB to pay, we know that expected net present value of the swap at origination must be zero, so

\[ \frac{5.75 - \bar{R}}{1 + \frac{1}{(0.08)}} + \frac{6.50 - \bar{R}}{1.10} = 0 \]

Solving the preceding equation, \( \bar{R} = 6.11 \). Hence, in the case of a market swap, the appropriate fixed rate is 12.22%, not the par rate of 5.