A digital filter is a minimum-phase filter if and only if all of its zeros lie inside or on the unit circle; otherwise, it is nonminimum-phase or mixed-phase.

The system and its inverse are both causal and BIBO stable.

The filter produces the minimum amount of group delay.
Group Delay

- Negative of the slope of the phase response of a linear system, i.e., filters

\[ D(\omega) = -\frac{d}{d\omega} (\angle H(\omega)) \quad \text{or} \quad D(f) = -\frac{1}{2\pi} \frac{d}{df} (\angle H(f)) \]

- The amount by which the spectral component at frequency \( f \) gets delayed as it is processed by the filter

- A digital linear-phase filter has a \textbf{constant} group delay except possibly at frequencies at which the magnitude response is zero

**Example**  Consider the following four second-order IIR filters. For each filter, (a) Generate the pole-zero plot, and (b) Plot both the magnitude and phase response.

\[
H_{00}(z) = \frac{2(z + 0.5)(z - 0.5)}{(z - 0.5)^2 + 0.25} \quad H_{10}(z) = \frac{(z + 2)(z - 0.5)}{(z - 0.5)^2 + 0.25}
\]
\[
H_{01}(z) = \frac{-(z + 0.5)(z - 2)}{(z - 0.5)^2 + 0.25} \quad H_{11}(z) = \frac{-0.5(z + 2)(z - 2)}{(z - 0.5)^2 + 0.25}
\]
Pole-zero Plot

Magnitude Response
FIR Filter Characteristics

- Completely specified by input-output relation:
  \[ y[n] = \sum_{k=0}^{M} b_k x[n - k] \]
- \( b_k \) = filter coefficients and \( M + 1 \) = filter length
- All poles are at the origin \( \rightarrow \) always stable
- Impulse response has only a finite number of terms \( \rightarrow \) finite length
Group Delay

- Negative of the slope of the phase response of a linear system, i.e., filters

\[ D(\omega) = -\frac{d}{d\omega} (\angle H(\omega)) \quad \text{or} \quad D(f) = \frac{1}{2\pi} \frac{d}{df} (\angle H(f)) \]

- The amount by which the spectral component at frequency \( f \) gets delayed as it is processed by the filter

- A digital linear-phase filter has a constant group delay except possibly at frequencies at which the magnitude response is zero

Generalized Linear-phase Filters

\[ H(\omega) = |H(\omega)| \exp\{\angle H(\omega)\} = |A_r(\omega)| \exp\{j(\alpha - \tau \omega)\} \]

where \( A_r(\omega) = \) Amplitude Response of \( H(z) \), \( \alpha = \) phase offset, and \( \tau = \) group delay

\[
\tan(\angle H(\omega)) = \tan(\alpha - \tau \omega) = \frac{\sin(\alpha - \tau \omega)}{\cos(\alpha - \tau \omega)} \quad (1)
\]

But, \( H(\omega) = \sum_{n=0}^{M} h[n] e^{-j\omega n} = \sum_{n=0}^{M} h[n] \cos\omega n - j \sum_{n=0}^{M} h[n] \sin\omega n \)

\[
\tan(\angle H(\omega)) = -\frac{\sum_{n=0}^{M} h[n] \sin(\omega n)}{\sum_{n=0}^{M} h[n] \cos(\omega n)} \quad (2)
\]
Generalized Linear-phase Filters

(1) = (2) \rightarrow \sum_{n=0}^{M} h[n] \sin(\alpha - \tau \omega) \cos \omega n = -\sum_{n=0}^{M} h[n] \cos(\alpha - \tau \omega) \sin \omega n

Using trigonometric identity: \sin(A + B) = \sin A \cos B + \cos A \sin B,

\rightarrow \sum_{n=0}^{M} h[n] \sin(\alpha + \omega(n - \tau)) = 0 \quad (3)

This is the necessary condition for \( h[n] \) to have linear phase.

Two possible cases:

1) \( \alpha = 0 \) or \( \pi \), \( \tau = \frac{M}{2} \), and \( h[n] = h[M - n] \) \quad (even symmetry)

2) \( \alpha = \frac{\pi}{2} \) or \( \frac{3\pi}{2} \), \( \tau = \frac{M}{2} \), \( h[n] = -h[M - n] \) \quad (odd symmetry)

Types of Linear-phase FIR filters

<table>
<thead>
<tr>
<th>Filter type</th>
<th>( h[n] ) symmetry</th>
<th>Filter order</th>
<th>Phase offset</th>
<th>End-point zeros</th>
<th>Candidate filters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Even</td>
<td>Even</td>
<td>0</td>
<td>None</td>
<td>All</td>
</tr>
<tr>
<td>2</td>
<td>Even</td>
<td>Odd</td>
<td>0</td>
<td>( z = -1 )</td>
<td>LP, BP</td>
</tr>
<tr>
<td>3</td>
<td>Odd</td>
<td>Even</td>
<td>( \pi/2 )</td>
<td>( z = \pm 1 )</td>
<td>BP</td>
</tr>
<tr>
<td>4</td>
<td>Odd</td>
<td>Odd</td>
<td>( \pi/2 )</td>
<td>( z = 1 )</td>
<td>HP, BP</td>
</tr>
</tbody>
</table>
Impulse Responses of Four Types of FIR Linear-phase Filters

\[ h[n] = n^2, \quad 0 \leq n \leq \left\lfloor M/2 \right\rfloor \]

**Linear-phase Zeros**

- Symmetry condition imposes constraints on the zeros of a linear-phase FIR filter
- Transfer function satisfies: \( H(z) = \pm z^{-M} H(z^{-1}) \)
- Type 1: end-point zeros possible
- Type 2: an end-point zero at \( z = -1 \)
- Type 3: end-point zeros at \( z = \pm 1 \)
- Type 4: an end-point zero at \( z = 1 \)
- All types: complex zeros in groups of four (\( r \neq 1 \))
Example  Construct a type 1 linear-phase filter of order 2 with Coefficients satisfying $|b_k| = 1, \forall k$. Also, find the transfer function and its zeros.
Example  Construct a type 2 linear-phase filter of order 1 with Coefficients satisfying $|b_k| = 1$, $\forall k$. Also, find the transfer function and its zeros.

Example  Construct a type 3 linear-phase filter of order 2 with Coefficients satisfying $|b_k| = 1$, $\forall k$. Also, find the transfer function and its zeros.
Example  Construct a type 4 linear-phase filter of order 1 with Coefficients satisfying $|b_k| = 1, \forall k$. Also, find the transfer function and its zeros.

The Windowing Method

- Start with the desired or ideal frequency response
- Compute IDTFT to obtain the desired impulse response according the filter type & order
- Truncate the resulting impulse response using one of the finite-length windowing functions, i.e., rectangular, Bartlett, Hamming, Hanning, and Blackman
Ideal Lowpass Characteristics

\[ x[n] = \frac{1}{2\pi} \int_{-W}^{W} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n} \]

\[ x[0] = \frac{1}{2\pi} \int_{-W}^{W} d\omega = \frac{W}{\pi} \]

Impulse Responses of Ideal Linear-phase type-1
FIR filters of Order \( M = 2\tau \)

<table>
<thead>
<tr>
<th>Filter type</th>
<th>( h[n] ), ( 0 \leq n \leq M, n \neq \tau )</th>
<th>( h[\tau] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lowpass</td>
<td>( \frac{\sin(\omega_c(n-\tau))}{\pi(n-\tau)} )</td>
<td>( \frac{\omega_c}{\pi} )</td>
</tr>
<tr>
<td>Highpass</td>
<td>( -\frac{\sin(\omega_c(n-\tau))}{\pi(n-\tau)} )</td>
<td>( \frac{\pi - \omega_c}{\pi} )</td>
</tr>
<tr>
<td>Bandpass</td>
<td>( \frac{\sin(\omega_h(n-\tau)) - \sin(\omega_l(n-\tau))}{\pi(n-\tau)} )</td>
<td>( \frac{\omega_h - \omega_l}{\pi} )</td>
</tr>
<tr>
<td>Bandstop</td>
<td>( \frac{\sin(\omega_l(n-\tau)) - \sin(\omega_h(n-\tau))}{\pi(n-\tau)} )</td>
<td>( \frac{\pi - (\omega_h - \omega_l)}{\pi} )</td>
</tr>
</tbody>
</table>
Example: Construct a type 1 linear-phase filter of order 6 with coefficients satisfying the highpass response characteristics and cutoff frequency of 2000 Hz assuming a sampling frequency of 8000 Hz. Also, find the transfer function and generate the pole-zero plot. Repeat for order 40.

Commonly-used Windowing Functions

1. Rectangular window
   
   $$w[n] = \begin{cases} 
   1, & 0 \leq n \leq M \\
   0, & \text{otherwise} 
   \end{cases}$$

2. Bartlett window
   
   $$w[n] = \begin{cases} 
   2n/M, & 0 \leq n \leq M/2 \\
   2 - 2n/M, & M/2 < n < M \\
   0, & \text{otherwise} 
   \end{cases}$$

3. Hanning window
   
   $$w[n] = \begin{cases} 
   0.5 - 0.5 \cos(2\pi n/M), & 0 \leq n \leq M \\
   0, & \text{otherwise} 
   \end{cases}$$

4. Hamming window
   
   $$w[n] = \begin{cases} 
   0.54 - 0.46 \cos(2\pi n/M), & 0 \leq n \leq M \\
   0, & \text{otherwise} 
   \end{cases}$$

5. Blackman window
   
   $$w[n] = \begin{cases} 
   0.42 - 0.5 \cos(2\pi n/M) + 0.08 \cos(4\pi n/M), & 0 \leq n \leq M \\
   0, & \text{otherwise} 
   \end{cases}$$
Meeting Design Specifications

- Appropriate window selected based on frequency-domain specifications
- Estimate the filter order, \( M \), to control the width of the normalized transition band of the filter.

\[ \Delta F = \frac{|F_T - F_P|}{F_s} \]

\[ F_C = \frac{F_p + F_T}{2} \]

MATLAB Implementation

- Function \( B = \text{firwd}(N, \text{Ftype}, \text{WL}, \text{WH}, \text{Wtype}) \)

- MATLAB user-defined function for FIR filter design using the windowing method (text, pp.288-290)

- Input Arguments:
  - \( N \) = number of filter taps (must be an odd number)
    = \( M+1 \) where \( M \) is a filter order (even number for Type 1)
  - \( \text{Ftype} \) = filter type (1 – lowpass, 2 – highpass, 3 – Bandpass, 4 – bandstop)
  - \( \text{WL} \) = lower cut-off frequency in rad (set to zero for highpass)
  - \( \text{WH} \) = upper cutoff frequency in rad (set to zero for lowpass)
  - \( \text{Wtype} \) = window type (1 – rectangular, 2 – triangular, 3 – Hanning, 4 – Hamming, 5 – Blackman)
Example 7.11 Design a type 1 linear-phase filter with coefficients satisfying bandstop response characteristics with the following specifications:

- Lower cutoff frequency of 1250 Hz
- Lower transition width of 1500 Hz
- Upper cutoff frequency of 2850 Hz
- Upper transition width of 1300 Hz
- Stopband attenuation of 60 dB
- Passband ripple of 0.02 dB
- Sampling frequency of 8000 Hz.

### Design Characteristics of Windows

<table>
<thead>
<tr>
<th>Window Type</th>
<th>Filter Order (M)</th>
<th>Passband Ripple</th>
<th>Stopband Attenuation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\delta_p$</td>
<td>$A_p$ (dB)</td>
</tr>
<tr>
<td>Rectangular</td>
<td>0.9/ΔF</td>
<td>0.0819</td>
<td>0.7416</td>
</tr>
<tr>
<td>Hanning</td>
<td>3.1/ΔF</td>
<td>0.0063</td>
<td>0.0546</td>
</tr>
<tr>
<td>Hamming</td>
<td>3.3/ΔF</td>
<td>0.0022</td>
<td>0.0194</td>
</tr>
<tr>
<td>Blackman</td>
<td>5.5/ΔF</td>
<td>0.00017</td>
<td>0.0017</td>
</tr>
</tbody>
</table>
FIR Comb Filters

- Based on a model for a single echo
- The transfer function is

\[ H(z) = 1 + \alpha z^{-R} \]

where \( R \) is the amount of delay in samples
- The block diagram representation is

- If \( \alpha = 1 \), this is either a type 1 or 2 linear-phase filters
Tremolo Effect

- A repetitive up/down variation in the volume of the original signal
- The block diagram representation is

\[
x[n] \quad \xrightarrow{1-\alpha} \quad + \quad \xrightarrow{\alpha \beta[n]} \quad y[n]
\]

- Depth of the effect controlled by \( \alpha \), where \( 0 \leq \alpha \leq 1 \)
- Variation in volume controlled by time-varying \( \beta \)

\[
\beta[n] = \frac{1}{2} \left[ 1 - \cos \left( \frac{2\pi F_0 n}{F_s} \right) \right], \quad F_0 = 7 \text{ Hz}
\]

Flanging Effect

- Based on an idea of adding echo to original signal
- The block diagram representation is

\[
x[n] \quad \xrightarrow{\alpha} \quad + \quad \xrightarrow{\beta[n]} \quad y[n]
\]

- Strength of echo controlled by \( \alpha \), where \( 0 \leq \alpha \leq 1 \)
- Delay pattern controlled by time-varying \( \beta \), e.g. a slow sinusoidally-varying delay from 0 to \( R \) samples

\[
\beta[n] = \frac{R}{2} \left[ \cos \left( \frac{2\pi F_0 n}{F_s} \right) \right], \quad F_0 = 1 \text{ Hz}
\]