Discrete-time Signals

ELEC 423
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Mathematical Representation

- \( x[n] \) represents a DT signal, i.e., a sequence of numbers defined only at integer values of \( n \) (undefined for noninteger values of \( n \))
- Each number \( x[n] \) is called a sample
- \( x[n] \) may be a sample from an analog signal
  \[
  x_d[n] = x_a(nT_s),
  \]
  where \( T_s = \) sampling period
Discrete-time Signals

- Discrete-time, continuous-valued amplitude (sampled-data signal)
- Discrete-time, discrete-valued amplitude (digital signal)
- In practice, we work with digital signals
Signal Transformations

Operations performed on the independent and the dependent variables

1) Reflection or Time Reversal or Folding
2) Time Shifting
3) Time Scaling
4) Amplitude Scaling
5) Amplitude Shifting

Note: The independent variable is assumed to be $n$ representing sample number.

Time reversal or Folding or Reflection

For DT signals, replace $n$ by $-n$, resulting in

$$y[n] = x[-n]$$

$y[n]$ is the reflected version of $x[n]$ about $n = 0$ (i.e., the vertical axis).
**Time Shifting (Advance or Delay)**

For DT Signals: replace $n$ by $n - n_0$
where $n_0$ is the amount of shift resulting in
\[
y[n] = x[n - n_0],
\]
\[
n_0 \in \mathbb{I}^+ \quad \Rightarrow \quad x[n] \text{ is shifted to the right by } n_0 \text{ (Delay)}
\]
\[
n_0 \in \mathbb{I}^- \quad \Rightarrow \quad x[n] \text{ is shifted to the left by } n_0 \text{ (Advance)}
\]

**Time Scaling**

**Sampling rate alteration**

Down-sampling: \[ y[n] = x[kn], \quad k \in I^+ \]
the sampling rate of $y[n]$ is $(1/k)^{th}$ that of $x[n]$

Up-sampling: \[ y[n] = \begin{cases} x[n/D], & \text{if } n/D \text{ is an integer} \\ 0, & \text{otherwise} \end{cases} \]
the sampling rate of $y[n]$ is $D$ times that of $x[n]$
Amplitude Scaling

For DT signals, multiply $x[n]$ by the scaling factor $A$, where $A$ is a real constant.

$$y[n] = Ax[n]$$

If $A$ is negative, $y[n]$ is the amplitude-scaled and reflected version of $x[n]$ about the horizontal axis.

Amplitude Shifting

For DT signals, add a shifting factor $A$ to $x[n]$, where $A$ is a real constant.

$$y[n] = x[n] + A$$
Signal Characteristics

- Deterministic vs. Random
- Finite-length vs. Infinite-length
- Right-sided/ Left-sided/ Two-sided
- Causal vs. Anti-causal
- Periodic vs. Aperiodic (Non-periodic)
- Real vs. Complex
- Conjugate-symmetric vs. Conjugate-antisymmetric
- Even vs. Odd

Even & Odd DT Signals

A complex-valued sequence $x[n]$ is said to be

- Conjugate symmetric if $x[-n] = x^*[n]$,
- Conjugate antisymmetric if $x[-n] = -x^*[n]$.

A real-valued signal $x[n]$ is said to be

- even if $x[-n] = x[n]$,
- odd if $x[-n] = -x[n]$. 
### DT Periodic Signals

A DT signal $x[n]$ is said to be periodic if

$$x[n] = x[n + N] = x[n + kN_0], \quad n, k \in I$$

$N_0$ is the smallest positive integer called the fundamental period (dimensionless)

$$\omega_0 = \frac{2\pi}{N_0} = \text{the fundamental angular frequency (radians)}$$

### DT Sinusoidal Signals

$$x[n] = A \cos[\omega_0 n + \phi]$$

$n$ is dimensionless (sample number)

$\omega_0$ and $\phi$ have units of radians

$x[n]$ may or may not be periodic

periodic if $\omega_0$ is a rational multiple of $2\pi$, i.e.,

$$\omega_0 = \frac{2\pi m}{N_0}, \quad m \in I \text{ and } N_0 \text{ the fundamental period}$$
Discrete-time periodic sinusoidal sequences for several different frequencies

An example of a discrete-time non-periodic sinusoidal sequence
DT Unit Impulse

- DT Unit impulse (Kronecker delta function)
  \[ \delta[n] = \begin{cases} 
    0, & n \neq 0 \\
    1, & n = 0 
  \end{cases} \]

- Properties:
  1) \[ \sum_{n=-\infty}^{\infty} \delta[n] = 1 \]
  2) \[ \delta[n] = \delta[-n] \]
  3) \[ x[n] \delta[n-n_0] = x[n_0] \delta[n-n_0] \]
  4) \[ \sum_{n=-\infty}^{\infty} x[n] \delta[n-n_0] = x[n_0] \]

Unit Step Function

\[ u[n] = \begin{cases} 
    0, & n < 0 \\
    1, & n \geq 0 
  \end{cases} \]

\( u[n] \) can be written as a sum of shifted unit impulses as

\[ u[n] = \sum_{k=0}^{\infty} \delta[n-k] \]
Exponential Signals

\[ x[n] = Br^n \]
where \( B \) and \( r \) are real or complex numbers

By letting \( r = e^\alpha \), \( \alpha \in \mathbb{R} \)

\[ x[n] = Be^{\alpha n} \]

Signal Metrics

- Energy
- Power
- Magnitude
- Area
- Average Value
- Zero Crossing
**Energy**

\[ E_x = \sum_{n} |x[n]|^2 \]

An infinite-length sequence with finite sample values may or may not have finite energy.

**Power**

\[ P_x = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} |x[n]|^2 \]

- A finite energy signal with zero average power is called an ENERGY signal.
- An infinite energy signal with finite average power is called a POWER signal.
Average Power

- Average over one period for periodic signal, e.g.,
\[ P_x = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} |x[n]|^2 \quad \text{for any } n_0 \]
- Root-mean-square value: \( x_{rms} = \sqrt{P_x} \)

Magnitude & Area

Magnitude: \( M_x = \max_n |x[n]| \)
Bounded if \( M_x < \infty \)

Area: \( A_x = \sum_n x[n] \)
Average value & ZCR

\[ x_{avg} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n] \]

ZCR Computation: Increment ZCR count by 1 if there is a sign change between two adjacent samples