Discrete Fourier Transform (2)

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Digital Frequency

\[ f = F T_s = \frac{F}{F_s} \quad \text{or} \quad \omega = \Omega T_s = \frac{\Omega}{F_s} \]

where \( T_s = \) sampling interval

\[ F_s = \text{sampling frequency} = \frac{1}{T_s} \]

\( X(e^{i\omega}) \) is simply a frequency-scaled version of \( X(j\Omega) \)

\( \rightarrow \) Normalization of the frequency axis so that

\( \Omega = \Omega_s \) in \( X(j\Omega) \) is normalized to \( \omega = 2\pi \) for \( X(e^{i\omega}) \)

Or \( F = F_s \) in \( X(F) \) is normalized to \( f = 1 \) for \( X(f) \)
Spectral Analysis via the DFT

- An important application of the DFT is to numerically determine the spectral content of signals.
- However, the extent to which this is possible is limited by two factors:
  1. truncation of the signal → causes leakage
  2. frequency domain sampling → causes picket fence effect & time-domain aliasing
Spectral Leakage

- Two theoretical assumptions about the sampled sequence when computing DFT:
  1) periodic
  2) continuous and band-limited to the folding freq.
- Spectral leakage = the presence of harmonics
- Caused by amplitude discontinuity in the sampled sequence resulting from signal truncation (i.e., a wrong choice of N) such that the 2\textsuperscript{nd} assumption is violated.

Amplitude Discontinuity

Truncation of a 32-point, 1-Hz sinusoidal sequence and the resulting periodic extension using N = 16 (top) and N = 18 samples (bottom)
Signal samples and spectra without (top) and with (bottom) spectral leakage

Signal Truncation & Spectral Leakage

let \( x_t(n) = x(n) \cdot w(n) \) \( t \) - truncated

where \( w(n) \) is a rectangular window

\[
w(n) = \begin{cases} 
1, & 0 \leq n \leq N-1 \\
0, & \text{else} 
\end{cases}
\]

\[
W(e^{j\omega}) = e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}
\]
And, \( x(n) = \cos(\omega_0 n) \)

\[
\cos(\omega_0 n) \xrightarrow{\text{DFT}} [\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)]
\]

By the product theorem,

\[
X_t(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j(\omega - \mu)}) X(e^{j\mu}) \, d\mu
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j(\omega - \mu)}) [\pi \delta(\mu - \omega_0) + \pi \delta(\mu + \omega_0)] \, d\mu
\]

\[
X_t(e^{j\omega}) = \frac{1}{2} [W(e^{j(\omega - \omega_0)}) + W(e^{j(\omega + \omega_0)})]
\]

Spectral Leakage caused by sidelobes of the ‘sinc’ function

Sidelobes may be suppressed at the expense of a wider mainlobe by using a smoothly tapering window.
Commonly-used Windowing Functions

1. Rectangular window
\[ w[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]

2. Bartlett window
\[ w[n] = \begin{cases} \frac{2n}{M}, & 0 \leq n \leq M/2 \\ 2 - \frac{2n}{M}, & M/2 < n < M \\ 0, & \text{otherwise} \end{cases} \]

3. Hanning window
\[ w[n] = \begin{cases} 0.5 - 0.5 \cos(\frac{2\pi n}{M}), & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]

4. Hamming window
\[ w[n] = \begin{cases} 0.54 - 0.46 \cos(\frac{2\pi n}{M}), & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]

5. Blackman window
\[ w[n] = \begin{cases} 0.42 - 0.5 \cos(\frac{2\pi n}{M}) + 0.08 \cos(\frac{4\pi n}{M}), & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]
Windowing operation: original sequence (top), window (middle), and windowed sequence (bottom) after pointwise multiplication

Spectral leakage reduction by windowing operation
Frequency Domain Sampling

**Time-domain sampling:** the choice of $T_s$ determines whether the undesirable aliasing in the frequency domain will occur.

**Frequency-domain sampling:** the choice of $N$ or the number of DFT points determines whether the undesirable aliasing in the time domain will occur. Also, the inability of the DFT to observe the spectrum as a continuous function causes the **picket-fence effect**.

Picket-fence Effect

Looking at an FFT spectrum is a little like looking at a mountain range through a picket fence.

In general, the peaks in an FFT spectrum will be measured too low in level, and the valleys too high.

Size of the picket determined by the frequency spacing: $\Delta F = \frac{F_s}{M}$
Picket-fence Effect

1) Spectral peak is midway between sample locations.
2) Each sidelobe peak occurs at a sample location.

- The effect is reduced by increasing the frequency spacing, i.e., increase $N$, the number of DFT points
- Zero-pad the original sequence to $M > N$ points
- $M$ must be a power of 2 if the Fast Fourier Transform (FFT) algorithm is used
- Choice of $M$ depends on the frequency components in the original sequence

Effect of Zero Padding

- The resulting spectrum is an interpolated version of the original spectrum, but with reduced frequency spacing
- Does not recover information lost by the time-domain sampling process
- Better detailed signal spectrum with a finer frequency resolution obtained by adding more data samples (i.e., longer sequence of data)
- Frequency resolution limited by the Raleigh Limit
Zero-padding effect on DFT calculations of a 12-pt sequence of an analog signal containing 10-Hz and 25-Hz sinusoidal components sampled at 100 Hz with: (A) no zero padding, (B) 4-point zero-padding, and (C) 16-point zero-padding.

Example: Consider a noise-free sequence with a single sinusoidal spectral component at 330.5 Hz and sampled at 1024 Hz. Determine the locations of the spectrum peaks for \( N = 256 \) and 2048 zero-padded DFT computational points. Also, sketch both spectra. Comment on the presence of spectral leakage and picket-Fence effect.
EX. Consider the signal

\[ x(t) = \cos(2\pi \cdot 50t) + 2\cos(2\pi \cdot 7000t) \]

which is sampled starting at time \( t = 0 \) with an ideal sampler (i.e., no pre-filter) operating at a 10 kHz rate. A 100-point DFT of the first 100 samples of this signal is computed.

(a) Determine the approximate values for \( k \) (in the range of 0 to 99) at which spectral peaks will be observed. Also, give the approximate amplitude of these peaks.

(b) For each spectral peaks in (a), state whether or not that peak will exhibit spectral leakage and picket-fence effect.
n = 0:99;
Fs = 10000;
x = \cos(2*\pi*50*n/Fs)+2*\cos(2*\pi*7000*n/Fs);
subplot(2,1,1)
stem(n,x); xlabel('n'); ylabel('x[n]');
subplot(2,1,2)
stem(n,abs(fft(x)));
xlabel('k'); ylabel('X(k)');
\[ x(n) = \cos(\omega_0 n) \]

\[ \cos(\omega_0 n) \stackrel{\text{DTFT}}{\leftrightarrow} [\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)] \]

\[ \omega_0 = \frac{2\pi k_0}{N} \]

\[ x(n) = \cos\left(2\pi k_0 \frac{n}{N}\right), \quad 0 \leq n \leq N-1 \]

\[ X(k) = \frac{N}{2} \left[ \delta(k - k_0) + \delta(k - (N - k_0)) \right], \quad 0 \leq k \leq N-1 \]
Example 4.6 & 4.7: A 1024-point DT sequence was obtained from sampling an analog signal at 10 kHz.
(a) Determine the frequency spacing if a 2048-pt DFT calculation with zero-padding is performed.
(b) It is required that the frequency resolution in the spectrum calculation without zero-padding be less than 0.5 Hz, determine the number of FFT data points needed.

Time-domain Aliasing

• Caused by periodic assumption of the time-domain sequence
• Regular → circular or cyclic convolution
• Implication on discrete-time system analysis
• Zero-pad the time-domain sequence such that
  \# of DFT points = L+P−1
  L, P are the lengths of the two sequences to be convolved
Example Consider the following two 4-point sequences:
\[ x[n] = \cos\left(\frac{\pi n}{2}\right), \quad n = 0, 1, 2, 3 \]
\[ h[n] = 2^n, \quad n = 0, 1, 2, 3 \]
Compute \( x[n]*h[n] \) by direct convolution and DFT methods.

Direct (Linear) Convolution

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Number of operations needed to perform convolution of two N-point sequences

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<th>N</th>
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<td>(N²)</td>
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