James Clerk Maxwell
Scottish Physicist & Mathematician
13 June 1831 – 5 November 1879
PRINCIPLE OF CONSERVATION OF ELECTRIC CHARGE

- The time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume.

- Mathematically, \( I_{\text{out}} = -\frac{d}{dt}(Q_{\text{encl}}) \) or
  \[
  \oint S \vec{J} \cdot d\vec{S} = -\frac{d}{dt} \int \rho_v dv = -\int v \frac{\partial \rho_v}{\partial t} dv
  \]

  Note that \( v \) is the volume enclosed by a closed surface \( S \) and \( \rho_v \) is assumed to be time-varying.

- Applying the divergence theorem to the LHS,
  \[
  \oint S \vec{J} \cdot d\vec{S} = \int (\nabla \cdot \vec{J}) dv = -\int v \frac{\partial \rho_v}{\partial t} dv
  \]
• Equating the integrands,

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \]

• The above equation represents the point form of the **continuity of current equation** and essentially states that there can be no accumulation of charge at any point.

• Note that for DC or steady currents, \( \rho_v \) is not time dependent.

\[ \nabla \cdot \mathbf{J} = 0 \]

→ The total charge leaving a volume (in terms of electric current) must be the same as the total charge entering it.

→ Kirchhoff’s current law (KCL) follows.
CONVECTION AND CONDUCTION CURRENTS

• Electric current is generally caused by the motion of electric charges.

• The current (in amperes) through a given surface is the amount of electric charges passing through that surface per unit time.

\[ I = \frac{dQ}{dt} \]

In terms of current density \( \vec{J} \) (current per unit area),

\[ I = \int_{S} \vec{J} \cdot d\vec{S} \]

• If such current flows through an insulating (non-conductive) medium, such as liquid, rarefied gas, or a vacuum, it is known as convection current.
Consider a flow of charge of density $\rho_v$ at velocity $\bar{u} = u_y\hat{y}$.

\[
\Delta I_y = \frac{\Delta Q}{\Delta t} = \frac{\rho_v\Delta y}{\Delta t} = \rho_v\Delta S \frac{\Delta y}{\Delta t} = \rho_v\Delta S u_y
\]

\[
J_y = \frac{\Delta I_y}{\Delta S} = \rho_v u_y
\]

In a general direction, the convection current density, $\bar{J}_{cv}$, (in A/m$^2$)

\[
\bar{J}_{cv} = \rho_v \bar{u}
\]
**Conduction current** involves the movement of electrons through conductive medium in response to an applied electric field.

It can be shown that **conduction current density** is given by

\[ \vec{J}_{cd} = \sigma \vec{E} \]

where \( \vec{E} \) is an applied electric field

\[ \sigma = \frac{ne^2 \tau}{m} \]

is the conductivity of the conductor

- \( n \) is the number of electron per volume
- \( e \) is the charge of an electron
- \( m \) is the mass of an electron
- \( \tau \) is the average time interval between collision of electrons

This relationship is known as the point form of Ohm’s law.
MODIFICATION TO AMPERE’S LAW

Recall Ampere’s law for magnetostatic filed:
\[ \nabla \times \vec{H} = \vec{J}_{cd} \quad \text{and} \quad \nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}_{cd} = 0 \]

In order to satisfy the continuity of current equation for time-varying fields, let
\[ \nabla \times \vec{H} = \vec{J}_{cd} + \vec{J}_d \quad \text{and} \quad \nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}_{cd} + \nabla \cdot \vec{J}_d = 0 \]

\[ \rightarrow \quad \nabla \cdot \vec{J}_d = -\nabla \cdot \vec{J}_{cd} = -\left( -\frac{\partial \rho_v}{\partial t} \right) = \frac{\partial}{\partial t} (\nabla \cdot \vec{D}) = \nabla \cdot \frac{\partial \vec{D}}{\partial t} \]

\[ \rightarrow \quad \vec{J}_d = \frac{\partial \vec{D}}{\partial t} \quad \text{and is called displacement current density} \]

The modified Ampere’s law is
\[ \nabla \times \vec{H} = \vec{J}_{cd} + \frac{\partial \vec{D}}{\partial t} \]
# Maxwell’s Equations for Time Varying Fields

<table>
<thead>
<tr>
<th>Integral form</th>
<th>Differential or Point form</th>
<th>Remarks</th>
</tr>
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<tbody>
<tr>
<td>$\oint \mathbf{E} \cdot d\mathbf{l} = - \int_S \left( \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S}$</td>
<td>$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}$</td>
<td>Faraday’s</td>
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<td>$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$</td>
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<td>$\int \mathbf{D} \cdot d\mathbf{S} = Q_{encl} = \int_{\nu} \rho_v d\nu$</td>
<td>$\nabla \cdot \mathbf{D} = \rho_v$</td>
<td>Gauss’s</td>
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<td>$\int \mathbf{B} \cdot d\mathbf{S} = 0$</td>
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### MAXWELL’S EQUATIONS FOR TIME-HARMONIC FIELDS

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Ex. Given that \( \vec{E} = E_m \cos(\omega t - \beta z)\hat{y} \) in free space, find \( \vec{D} \), \( \vec{B} \), and \( \vec{H} \). Also, sketch \( \vec{E} \) and \( \vec{H} \) on the same axes at \( t = 0 \).