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ELECTRIC FLUX DENSITY

- Electric flux ($\Psi_E$) is a scalar quantity measured in Coulomb and used in expressing the strength of a field of electric force in a given area. (True for fluids, particles, or energy in general)

- Electric flux density ($\vec{D}$) is a vector quantity measured in C/m$^2$ representing the rate of flow of electric flux (amount of electric flux per unit of cross-sectional area)

- Mathematically,  
  $$\vec{D} = \frac{d\Psi_E}{dS} \vec{a} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_o \vec{E}$$  
  where $\vec{a}$ is the direction flux lines (same as that of the field)

- Dot-multiplying both sides by $d\vec{S}$ and integrating over the surface defined by $d\vec{S}$,  
  $$\int_S \vec{D} \cdot d\vec{S} = \int_S \frac{d\Psi_E}{dS} \vec{a} \cdot d\vec{S} = \Psi_E$$
GRAPHICAL REPRESENTATION OF FIELDS

• Use directed line segments called flux lines to represent the strength and direction of the general tendency to flow

• The magnitude at a given point depicted either by the density or by the length of flux lines
GAUSS’S LAW

The total electric flux, $\Psi_E$, through any closed surface is equal to the total charge enclosed by that surface.

$$\Psi_E = \oint \vec{D} \cdot d\vec{S} = Q_{encl} = \int \rho_v dv$$  \hspace{1cm} \text{(Integral form)}

Applying divergence theorem,

$$\oint \vec{D} \cdot d\vec{S} = \int (\nabla \cdot \vec{D}) dv$$

$$\Rightarrow \quad \nabla \cdot \vec{D} = \rho_v$$  \hspace{1cm} \text{(Point or Differential form)}

The volume charge density at a given point in space is equal to the divergence of the electric flux density at the point.
• Gauss’s law is an alternative statement of Coulomb’s law through the application of divergence theorem.

• In SI units, one line of electric flux emanates from +1C and terminates on −1C.

• Gauss’s law provides an easy means of finding $\vec{E}$ or $\vec{D}$ for symmetrical charge distributions.

• Cartesian coordinates:

$$\nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

• Cylindrical coordinates:

$$\nabla \cdot \vec{D} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho D_\rho) + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

• Spherical coordinates:

$$\nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (D_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial D_\phi}{\partial \phi}$$
The divergence of a vector field $\vec{D}$ at a given point $P$ is the outward flow of flux per unit volume as the volume shrinks about $P$.

$$\text{div } \vec{D} = \nabla \cdot \vec{D} = \lim_{\Delta V \to 0} \frac{\iint_S \vec{D} \cdot d\vec{S}}{\Delta V}$$

**DIVERGENCE**

Positive divergence  negative divergence  zero divergence
Ex. Determine the charge density due to each of the following electric flux densities:

(a) \( \bar{D} = x^2 yz \hat{x} + xz \hat{z} \)

(b) \( \bar{D} = \rho \sin \phi \hat{\rho} + \rho^2 z \hat{\phi} + z \cos \phi \hat{z} \)

(c) \( \bar{D} = \frac{1}{r^2} \cos \theta \hat{r} + r \sin \theta \cos \phi \hat{\theta} + \cos \theta \hat{\phi} \)
APPLICATIONS OF GAUSS’S LAW

1. Examine whether symmetric charge distribution exists

2. If so, construct a mathematical closed surface (called Gaussian surface) such that $\vec{D}$ is normal or tangential to the surface.
   
   Normal $\Rightarrow \vec{D} \cdot d\vec{S} = D \, dS$
   
   Tangential $\Rightarrow \vec{D} \cdot d\vec{S} = 0$

3. Gaussian surface should have some of the symmetry exhibited by the charge distribution.
The vector field \( \vec{D} = D_r \hat{r} \) is everywhere normal to the spherical Gaussian surface.

\[
Q = Q_{encl} = \int_S \vec{D} \cdot d\vec{S} = \int \int D_r \hat{r} \cdot r^2 \sin \theta \, d\theta \, d\phi \, \hat{r} = D_r (4\pi r^2)
\]

\[
\vec{D} = \frac{Q}{4\pi r^2} \hat{r}
\]
INFINITE LINE CHARGE

\[ \vec{D} = D \rho \hat{\rho} \] is everywhere normal to the side surface of the cylindrical Gaussian surface.

\[ Q = Q_{encl} = \int \vec{D} \cdot d\vec{S} \]

\[ = \int_{top} + \int_{bottom} + \int_{side} \vec{D} \cdot d\vec{S} \]

\[ = \int \int D \rho \hat{\rho} \cdot \rho d\phi dz \hat{\rho} \]

\[ = D \rho (2\pi \rho l) \]

\[ \vec{D} = \frac{Q}{2\pi \rho l} \hat{\rho} = \frac{\rho_L}{2\pi \rho} \hat{\rho} \]
INFINITE SHEET OF CHARGE

Gaussian surface: a rectangular box cut symmetrically by the sheet of charge and has two faces parallel to the sheet

\[ \vec{D} = D_z \hat{z} \] normal to the top and bottom surfaces of the Gaussian surface

\[ Q = Q_{encl} = \oint_S \vec{D} \cdot d\vec{S} \]

\[ \rho_S A = D_z \left( \int_{top} dS + \int_{bottom} dS \right) = D_z (A + A) \]
UNIFORMLY CHARGED SPHERE

Gaussian surface: a spherical shell

\[ \vec{D} = \begin{cases} 
\frac{r}{3} \rho_v \hat{r} & 0 < r \leq a \\
\frac{a^3}{3r^2} \rho_v \hat{r} & r \geq a 
\end{cases} \]
Ex. A charge distribution with spherical symmetry has density

\[ \rho_v = \begin{cases} \frac{\rho_0 r}{R}, & 0 \leq r \leq R \\ 0, & r > R \end{cases} \]

Determine the electric field everywhere.
Ex. Given that $\vec{D} = \rho z \cos^2 \phi \hat{z}$ C/m², calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \leq z \leq 2$ m.