Continuous-time Fourier Series

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Orthogonal Expansion of CT Signals

- A linear combination of weighted orthogonal basis functions

\[ x(t) = \sum_{k} X_k \phi_k(t) \]

where \( X_k ' s = \) coefficients of expansion (weight)
Orthogonal Basis Functions

\{\phi_k(t), k \in I \} is a set of orthogonal basis functions

Orthogonality condition:

\[ <\phi_1(t), \phi_j(t)> = \int_a^b \phi_1(\tau)\phi_j^*(\tau)d\tau = \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases} \]

If \( \lambda_i = 1, \) \( \Rightarrow \) orthonormal
Periodic Complex Exponentials

\[ \phi_k(t) = e^{j k \Omega_0 t} = e^{j k \left( \frac{2 \pi}{T_0} \right) t} , \quad k = 0, \pm 1, \pm 2, \ldots \]

form a set of harmonically related complex exponentials, each being periodic with period \( T_0 = \frac{2 \pi}{\Omega_0} \).

- \( k = 0 \Rightarrow \) constant
- \( k = \pm 1 \Rightarrow \) fundamental components or 1\(^{st}\) harmonic
- \( k = \pm 2 \Rightarrow 2^{nd} \) harmonic
  
  \[ \vdots \]
- \( k = \pm N \Rightarrow N^{th} \) harmonic
**Continuous-time Fourier Series**

- A linear combination of harmonically-related complex exponentials

\[
\hat{x}(t) = \sum_{k} X_k e^{j k \Omega_0 t} = \sum_{k} X_k e^{j k \left(\frac{2\pi}{T_0}\right) t}
\]

where \( X_k \)'s are the real or complex coefficients of expansion

- \( \hat{x}(t) \) is an approximation or estimate of the given periodic signal \( x(t) \)
Fourier Series Approximation of a Periodic Square Wave

N = 1

N = 5

N = 10

N = 50

N = 100

N = 1000
Questions about CTFS

- How good is $\hat{x}(t)$ in approximating $x(t)$?
- Is it possible to obtain an exact representation of $x(t)$ in the form of $\hat{x}(t)$?
  - # of basis functions (harmonics) needed
  - convergence of Fourier Series with an infinite # of harmonics (i.e., infinite series)
- How does one obtain the coefficients?
Minimum Mean Square Error (MMSE)

- Let $\hat{x}_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t}$ be a Fourier series approximation of a given periodic signal $x(t)$.
- Define the error signal as $e_N(t) = x(t) - \hat{x}_N(t)$ and
  $$E_N = \int_{a}^{b} |e_N(t)|^2 dt ; a \leq t \leq b$$ as the mean square error.
- In order for $\hat{x}_N(t)$ to be an efficient approximation of $x(t)$, $E_N$ must be as small as possible over $[a,b] \Rightarrow MMSE$.
Fourier series coefficients

- It can be shown that MMSE is achieved when

\[ X_k = \frac{1}{T_0} \int_{0}^{T_0} x(t) e^{-j\Omega_0 t} dt \]

- A measure of the portion of the signal \(x(t)\) at each harmonic of the fundamental component

- \(X_0 = \frac{1}{T_0} \int_{0}^{T_0} x(t) dt\) = average value of \(x(t)\) over one period = dc or constant component of \(x(t)\)
Complex or Exponential Form

\[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} = \sum_{k=-\infty}^{\infty} X_k e^{jk\left(\frac{2\pi}{T_0}\right)t} \]

where \( X_0 = \frac{1}{T_0} \int_{T_0} x(t)dt \) = DC or Average value

\[ X_k = \frac{1}{T_0} \int_{T_0} x(t)e^{-jk\Omega_0 t} dt, \quad -\infty < k < \infty \]
Properties of CTFS

1) For real $x(t)$, $X_{-k} = X^*_k$, $\text{Re}\{X_k\} = \text{Re}\{X_{-k}\}$, $\text{Im}\{X_k\} = -\text{Im}\{X_{-k}\}$, $|X_k| = |X_{-k}|$, $\angle X_k = -\angle X_{-k}$

2) For real and even $x(t)$, $X_k$ real and even

3) For real and odd $x(t)$, $X_k$ purely imaginary and odd

4) Linearity: $Ax(t) + By(t) \xrightarrow{\text{CTFS}} AX_k + BY_k$

5) Parseval's relation: $\frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |X_k|^2$
Parseval’s Relation

- The total average power in a continuous-time periodic signal equals the sum of the average powers in all of its harmonic components

\[ \frac{1}{T} \int_{T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2 \]

where \( T \) is the period of \( x(t) \)
Example 1

Find the complex Fourier series coefficients of the signal $x(t) = \sin \Omega_0 t$
Example 2

Find the complex Fourier series coefficients of the signal

\[ x(t) = 1 + \sin \Omega_0 t + 2 \cos \Omega_0 t + \cos(2\Omega_0 t + \frac{\pi}{4}) \]
Example 3

Find the period of the following periodic signal and determine which harmonics are present in its Fourier series representation

\[ x(t) = \cos\left(\frac{2}{3} t + 30^\circ\right) + \sin\left(\frac{4}{5} t + \frac{\pi}{4}\right) \]
Example 4

- Find the complex Fourier series coefficients of the signal

\[
x(t) = \begin{cases} 
    A, & |t| < \frac{\tau}{2} \\
    0, & \frac{\tau}{2} < |t| < \frac{T}{2}
\end{cases}
\]
Example 4: Line Spectrum
Example 5

Verify Parseval’s theorem for the periodic sawtooth waveform whose first cycle is given by $x(t) = 5t$ for $0 \leq t \leq 4s$. 
Dirichlet Conditions

1) Over any period, $x(t)$ must be absolutely integrable:
   \[ \int_0^T |x(t)| \, dt < \infty \]

2) In any finite interval of time, $x(t)$ is of bounded variation (finite # of maxima and minima during a single period)

3) In any finite interval of time, there are only a finite number of finite discontinuities
Examples of periodic signals that violate Dirichlet's
Gibbs phenomenon

- Waveform of CTFS approximation ($N$-term truncation) exhibits ripples on both sides of discontinuities
- Ripple = overshoot and undershoot
- Number of ripples increases as $N$ increases
- The peak value of the ripples remains at 9% of the step size regardless of how big $N$ is
Figure 5.7 Truncated Fourier series expansions of the periodic square wave: (a) $N = 3$, (b) $N = 10$, and (c) $N = 30$. 
Trigonometric Form

\[ \hat{x}(t) = a_0 + \sum_{k=1}^{\infty} \{a_k \cos(k\Omega_0 t) + b_k \sin(k\Omega_0 t) \} \]

where \[ a_0 = X_0 \]
\[ a_k = X_k + X_{-k} \]
\[ b_k = j(X_k - X_{-k}) \]

Note: Only positive values of \( k \) and no \( b_0 \)
\[ \hat{x}(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k \Omega_0 t - \theta_k) \]

where \[ c_k = \sqrt{a_k^2 + b_k^2} \]

Only positive values of \( k \)

\begin{equation}
\begin{aligned}
\theta_k &= \begin{cases} 
\tan^{-1} \left( \frac{b_k}{a_k} \right), & a_k > 0 \\
\tan^{-1} \left( \frac{b_k}{a_k} \right) + \pi, & a_k < 0 
\end{cases}
\end{aligned}
\end{equation}