Key Equations for Waiting Lines

\( \lambda = \) average number of arrivals per time period (e.g., per hour)
\( \mu = \) average number of people or items served per time period

\[
(P-1) \hspace{1cm} P(X) = \frac{e^{-\lambda} \lambda^x}{x!} \hspace{1cm} \text{for} \hspace{0.2cm} X = 0, 1, 2, \ldots
\]

Poisson probability distribution used in describing arrivals.

\[
(P-2) \hspace{1cm} P(\text{service time} > t) = e^{-\mu t}, \hspace{0.2cm} \text{for} \hspace{0.2cm} t \geq 0
\]

Exponential probability distribution used in describing service times.

*Equations D-3 through D-9 describe the operating characteristics in a single-server queuing system that has Poisson arrivals and exponential service times.*

\[
(D-3) \hspace{1cm} p = \frac{\lambda}{\mu}
\]

Average server utilization in the system.

\[
(D-4) \hspace{1cm} L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}
\]

Average number of customers or units waiting in line for service.

\[
(D-5) \hspace{1cm} L = L_q + \frac{\lambda}{\mu}
\]

Average number of customers or units in the system.

\[
(D-6) \hspace{1cm} W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)}
\]

Average time a customer or unit spends waiting in line for service.
(D-7) \( W = W_q + 1/\mu \)

Average time a customer or unit spends in the system.

(D-8) \( P_0 = 1 - \lambda/\mu \)

Probability that there are zero customers or units in the system.

(D-9) \( P_n = (\lambda/\mu)^n P_0 \)

Probability that there are \( n \) customers or units in the system.

(D-10) Total cost = \( C_w \times L + C_s \times s \)

Total cost is the sum of waiting cost and service cost.

*Equations D-11 through D-18 describe the operating characteristics in a multiple-server queuing system that has Poisson arrivals and exponential service times.*

(D-11) \( p = \lambda/(s\mu) \)

Average server utilization in the system.

(D-12) \( P_0 = \frac{1}{\sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{\lambda}{\mu} \right)^i + \frac{1}{(s-1)!} \frac{\lambda}{\mu} \frac{\mu}{(s\mu - \lambda)} P_0} \)

Probability that there are zero customers or units in the system.

(D-13) \( L_q = \frac{(\lambda/\mu)^s \lambda \mu}{(s - 1)! (s\mu - \lambda)^2 P_0} \)

Average number of customers or units waiting in line for service.

(D-14) \( L = L_q + \lambda/\mu \)

The average number of customers or units in the system.
(D-15) \[ W_q = \frac{L_q}{\lambda} \]

Average time a customer or unit spends waiting in line for service.

(D-16) \[ W = W_q + \frac{1}{\mu} \]

Average time a customer or unit spends in the system.

(D-17) \[ P_n = \frac{(\frac{\lambda}{\mu})^n}{n!} P_0 \quad \text{for} \ n \leq s \]

Probability that there are \( n \) customers or units in the system, for \( n \leq s \).

(D-18) \[ P_n = \frac{(\frac{\lambda}{\mu})^n}{s! s^{(n-s)}} P_0 \quad \text{for} \ n > s \]

Probability that there are \( n \) customers or units in the system, for \( n > s \).

Equations D-19 through D-24 describe the operating characteristics in a single-server queuing system that has Poisson arrivals and constant service times.

(D-19) \[ p = \frac{\lambda}{\mu} \]

Average server utilization in the system.

(D-20) \[ L_q = \frac{\lambda^2}{2\mu(\mu - \lambda)} \]

Average number of customers or units waiting in line for service.

(D-21) \[ L = L_q + \frac{\lambda}{\mu} \]

Average number of customers or units in the system.

(D-22) \[ W_q = \frac{L_q}{\lambda} = \frac{\lambda}{2\mu(\mu - \lambda)} \]

Average time a customer or unit spends waiting in line for service.
(D-23) \[ W = W_q + 1/\mu \]

Average time a customer or unit spends in the system.

(D-24) \[ P_0 = 1 - \lambda/\mu \]

Probability that there are zero customers or units in the system.

Equations D-25 through D-30 describe the operating characteristics in a single-server queuing system that has Poisson arrivals and general service times.

(D-25) \[ p = \lambda/\mu \]

Average server utilization in the system.

(D-26) \[ L_q = \frac{\lambda^2 \sigma^2 + (\lambda/\mu)^2}{2(1-(\lambda/\mu))} \]

Average number of customers or units waiting in line for service.

(D-27) \[ L = L_q + \lambda/\mu \]

Average number of customers or units in the system.

(D-28) \[ W_q = L_q / \lambda \]

Average time a customer or unit spends waiting in line for service.

(D-29) \[ W = W_q + 1/\mu \]

Average time a customer or unit spends in the system.

(D-30) \[ P_0 = 1 - \lambda/\mu \]

Probability that there are zero customers or units in the system.
Equations D-31 through D-38 describe the operating characteristics in a multiple-server queuing system that has Poisson arrivals, exponential service times, and a finite population of size $N$.

\begin{equation}
\begin{align*}
P_0 &= \frac{1}{\sum_{n=0}^{s-1} \frac{N!}{(N-n)!n!} \left( \frac{\lambda}{\mu} \right)^n + \sum_{n=s}^{N} \frac{N!}{(N-n)!s!(s-n)!} \left( \frac{\lambda}{\mu} \right)^n} \\
\text{(D-31)}
\end{align*}
\end{equation}

Probability that there are zero customers or units in the system.

\begin{equation}
\begin{align*}
P_n &= \frac{N!}{(N-n)!n!} \left( \frac{\lambda}{\mu} \right)^n P_0, \quad \text{if } 0 \leq n \leq s \\
\text{(D-32)}
\end{align*}
\end{equation}

Probability that there are exactly $n$ customers in the system, for $0 \leq n \leq s$.

\begin{equation}
\begin{align*}
P_n &= \frac{N!}{(N-n)!s!(s-n)!} \left( \frac{\lambda}{\mu} \right)^n P_0, \quad \text{if } s < n \leq N \\
\text{(D-33)}
\end{align*}
\end{equation}

Probability that there are exactly $n$ customers in the system, for $s \leq n \leq N$.

\begin{equation}
\begin{align*}
P_n &= 0, \quad \text{if } n > N \\
\text{(D-34)}
\end{align*}
\end{equation}

Probability that there are exactly $n$ customers in the system, for $n > N$.

\begin{equation}
\begin{align*}
L_q &= \sum_{s=0}^{N-s} (n-s)P_n \\
\text{(D-35)}
\end{align*}
\end{equation}

Average number of customers or units waiting in line for service.

\begin{equation}
\begin{align*}
L &= \sum_{n=0}^{s-1} nP_n + L_q + s \left( 1 - \sum_{n=0}^{s-1} P_n \right) \\
\text{(D-36)}
\end{align*}
\end{equation}

Average number of customers or units in the system.
(D-37) \[ W_q = \frac{L_q}{\lambda (N - L)} \]

Average time a customer or unit spends waiting in line for service.

(D-38) \[ W = \frac{L}{\lambda (N - L)} \]

Average time a customer or unit spends in the system.